Chapter 1
The Characteristic Matrix of Nonuniqueness for First-Kind Equations

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1.1 Introduction

Let S be a finite domain in $\mathbb{R}^2$, bounded by a simple, closed, C^2 curve ∂S. We denote by x and y generic points in $S \cup \partial S$ and by $|x-y|$ the distance between x and y in the Cartesian metric. Also, let $C^{0,\alpha}(\partial S)$ and $C^{1,\alpha}(\partial S)$, $\alpha \in (0,1)$, be, respectively, the spaces of Hölder continuous and Hölder continuously differentiable functions on ∂S. In what follows, Greek and Latin indices take the values $1,2$ and $1,2,3$, respectively, and a superscript T denotes matrix transposition.

For any function f continuous on ∂S, we define the 'calibration' functional p by

$$pf = \int\limits_{\partial S} f\,ds.$$

Using the fundamental solution for the two-dimensional Laplacian

$$g(x,y) = -\frac{1}{2\pi}\ln|x-y|,$$

we define the single-layer harmonic potential of density φ by

$$(V\varphi)(x) = \int\limits_{\partial S} g(x,y)\varphi(y)\,ds(y).$$

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The proof of the following assertion can be found, for example, in [Co94] or [Co00].

Theorem 1. *For any* $\alpha \in (0,1)$*, there are a unique nonzero function* $\Phi \in C^{0,\alpha}(\partial S)$ *and a unique number* ω *such that*

$$V\Phi = \omega \quad \text{on } \partial S, \quad p\Phi = 1.$$

It is easy to see that Φ and ω depend on g and ∂S.

The numbers $2\pi\omega$ and $e^{-2\pi\omega}$ are called *Robin's constant* and the *logarithmic capacity* of ∂S.

For a circle with the center at the origin and radius R, both Φ and ω can be determined explicitly:

$$\Phi = \frac{1}{2\pi R}, \quad \omega = -\frac{1}{2\pi}\ln R.$$

For other boundary curves, Φ and ω are practically impossible to determine analytically and must be computed by numerical methods.

If the solution of the Dirichlet problem in S with data function f on ∂S is sought as $u = V\varphi$, then φ is a solution of the (weakly singular) first-kind boundary integral equation

$$V\varphi = f \quad \text{on } \partial S.$$

This is a well-posed problem if and only if $\omega \neq 0$. If $\omega = 0$, the above equation has infinitely many solutions, which are expressed in terms of Φ.

1.2 Plane Elastic Strain

Consider a plate made of a homogeneous and isotropic material with Lamé constants λ and μ which undergoes deformations in the (x_1, x_2)-plane. If the body forces are negligible, then its (static) displacement vector $u = (u_1, u_2)^{\mathrm{T}}$ satisfies the equilibrium system of equations [Co00]

$$Au = 0 \quad \text{in } S,$$

where

$$A(\partial_1, \partial_2) = \begin{pmatrix} \mu\Delta + (\lambda+\mu)\partial_1^2 & (\lambda+\mu)\partial_1\partial_2 \\ (\lambda+\mu)\partial_1\partial_2 & \mu\Delta + (\lambda+\mu)\partial_2^2 \end{pmatrix}.$$

It is not difficult to show [Co00] that the columns $F^{(i)}$ of the matrix

$$F = \begin{pmatrix} 1 & 0 & x_2 \\ 0 & 1 & -x_1 \end{pmatrix}$$

form a basis for the space of rigid displacements.

The 'calibrating' vector-valued functional p is defined for continuous 2×1 vector functions f by

$$pf = \int_{\partial S} F^{\mathrm{T}} f\, ds.$$

A matrix of fundamental solutions for A is [Co00]

$$D(x,y) = -\frac{1}{4\pi\mu(\gamma+1)} \times \begin{pmatrix} 2\gamma \ln|x-y| + 2\gamma + 1 - \dfrac{2(x_1-y_1)^2}{|x-y|^2} & -\dfrac{2(x_1-y_1)(x_2-y_2)}{|x-y|^2} \\ -\dfrac{2(x_1-y_1)(x_2-y_2)}{|x-y|^2} & 2\gamma \ln|x-y| + 2\gamma + 1 - \dfrac{2(x_2-y_2)^2}{|x-y|^2} \end{pmatrix},$$

$$\gamma = \frac{\lambda + 3\mu}{\lambda + \mu}.$$

The single-layer potential of density φ is defined by

$$(V\varphi)(x) = \int_{\partial S} D(x,y)\varphi(y)\, ds(y).$$

The proof of the following assertion can be found in [Co00].

Theorem 2. *There is a unique* 2×3 *matrix function* $\Phi \in C^{0,\alpha}(\partial S)$ *and a unique* 3×3 *constant symmetric matrix* $\mathscr{C}$ *such that the columns* $\Phi^{(i)}$ *of* Φ *are linearly independent and*

$$V\Phi = F\mathscr{C} \quad \text{on } \partial S, \quad p\Phi = I,$$

where I *is the identity matrix.*

Clearly, Φ and $\mathscr{C}$ depend on A, D, and ∂S.

In the so-called alternative indirect method [Co00], the solution of the Dirichlet problem in S with data function f on ∂S is sought in the form

$$u = V\varphi. \tag{1.1}$$

Then the problem reduces to the (weakly singular) boundary integral equation

$$V\varphi = f \quad \text{on } \partial S. \tag{1.2}$$

Theorem 3. *Equation* (1.1) *has a unique solution* $\varphi \in C^{0,\alpha}(\partial S)$, $\alpha \in (0,1)$, *for any* $f \in C^{1,\alpha}(\partial S)$ *if and only if* $\det\mathscr{C} \neq 0$. *In this case,* (1.2) *is the unique solution of the Dirichlet problem.*

If $\det\mathscr{C} = 0$, then the unique solution of the Dirichlet problem is obtained by solving an ill-posed modified boundary integral equation whose infinitely many solutions are constructed with Φ and $\mathscr{C}$.

In the so-called refined indirect method [Co00], the solution of the Dirichlet problem is sought as a pair $\{\varphi, c\}$ such that

$$u = V\varphi - Fc \quad \text{in } S, \quad p\varphi = s,$$

where s a constant 3×1 vector chosen (arbitrarily) a priori and c is a constant 3×1 vector. This leads to the system of boundary integral equations

$$V\varphi - Fc = f \quad \text{on } \partial S, \quad p\varphi = s. \tag{1.3}$$

Theorem 4. *System* (1.3) *has a unique solution* $\{\varphi, c\}$ *with* $\varphi \in C^{0,\alpha}(\partial S)$ *for any* $f \in C^{1,\alpha}(\partial S)$, $\alpha \in (0,1)$, *and any s.*

It is important to evaluate the arbitrariness in the representation of the solution with respect to the prescribed ‘calibration’ s.

Theorem 5. *If* $\{\varphi^{(1)}, c^{(1)}\}$, $\{\varphi^{(2)}, c^{(2)}\}$ *are two solutions of* (1.3) *constructed with* $s^{(1)}$ *and* $s^{(2)}$, *respectively, then*

$$\varphi^{(2)} = \varphi^{(1)} + \Phi(s^{(2)} - s^{(1)}),$$
$$c^{(2)} = c^{(1)} + \mathscr{C}(s^{(2)} - s^{(1)}).$$

It is not easy to compute Φ and $\mathscr{C}$ analytically, or even numerically, in arbitrary domains S, but this can be accomplished if S is a circular disk. Let ∂S be the circle with center at the origin and radius R. In this case, Φ and $\mathscr{C}$ can be determined analytically as

$$\Phi = \frac{1}{2\pi R}\begin{pmatrix} 1 & 0 & R^{-2}x_2 \\ 0 & 1 & -R^{-2}x_1 \end{pmatrix},$$

$$\mathscr{C} = -\frac{1}{4\pi\mu(\lambda+2\mu)R^2} \times \begin{pmatrix} (\lambda+3\mu)R^2(\ln R+1) & 0 & 0 \\ 0 & (\lambda+3\mu)R^2(\ln R+1) & 0 \\ 0 & 0 & -(\lambda+\mu) \end{pmatrix}.$$

Clearly, $\det\mathscr{C} = 0$ if and only if $R = e^{-1}$.

Analytic computation of Φ and $\mathscr{C}$ is practically impossible for non-circular domains, and must be performed numerically.

We choose four 3×1 constant vectors $s^{(0)}$, $s^{(i)}$ such that the set $\{s^{(i)} - s^{(0)}\}$ is linearly independent, and form the 3×3 matrix Σ with columns $s^{(i)} - s^{(0)}$. Also, we choose an arbitrary function f. Next, we compute the solutions $\{\varphi^{(0)}, c^{(0)}\}$, $\{\varphi^{(i)}, c^{(i)}\}$ of (1.3) corresponding to $s^{(0)}$, $s^{(i)}$, respectively, and f by the refined indirect method, then form the 2×3 matrix function Ψ with columns $\varphi^{(i)} - \varphi^{(0)}$ and the constant 3×3 matrix Γ with columns $c^{(i)} - c^{(0)}$.

From Theorem 4 it follows that

$$\varphi^{(i)} - \varphi^{(0)} = \Phi(s^{(i)} - s^{(0)}),$$
$$c^{(i)} - c^{(0)} = \mathscr{C}(s^{(i)} - s^{(0)}),$$

or, what is the same,

$$\Phi\Sigma = \Psi, \quad \mathscr{C}\Sigma = \Gamma;$$

hence,

$$\Phi = \Psi\Sigma^{-1}, \quad \mathscr{C} = \Gamma\Sigma^{-1}.$$

A similar analysis can be performed for other two-dimensional linear elliptic systems with constant coefficients—for example, the system modeling bending of elastic plates with transverse shear deformation [Co00]. No apparent connection exists between the matrix $\mathscr{C}$ and the characteristic constant ω of ∂S.

1.3 Numerical Examples

Consider a steel plate with scaled Lamé coefficients

$$\lambda = 11.5, \quad \mu = 7.69,$$

and let ∂S (see Fig. 1.1) be the curve of parametric equations

$$x_1(t) = 2\cos(\pi t) - \tfrac{4}{3}\cos(2\pi t) + \tfrac{10}{3},$$
$$x_2(t) = 2\sin(\pi t) + 2, \quad 0 \le t \le 2.$$

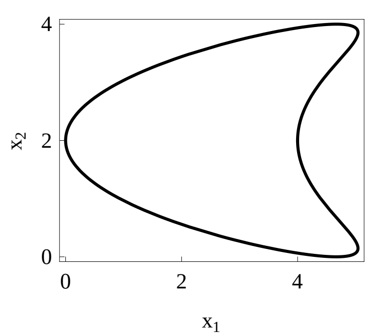

Fig. 1.1 The boundary curve ∂S.

We choose the vectors

$$s^{(0)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad s^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad s^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad s^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$f(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The approximating functions for computing $\varphi^{(0)}(t)$ and $\varphi^{(i)}(t)$ are piecewise cubic Hermite splines on 12 knots; that is, the interval $0 \le t \le 2$ is divided into 12 equal subintervals. Then the characteristic matrix (with entries rounded off to 5 decimal places) is

$$\mathscr{C} = \begin{pmatrix} -0.01627 & -0.01083 & -0.00370 \\ -0.01083 & -0.00892 & 0.00542 \\ -0.00370 & 0.00542 & 0.00185 \end{pmatrix}.$$

Here,

$$\det \mathscr{C} = 1.08273 \times 10^{-6}.$$

The graphs of the components $\Phi_{\alpha i}$ of Φ are shown in Fig. 1.2.

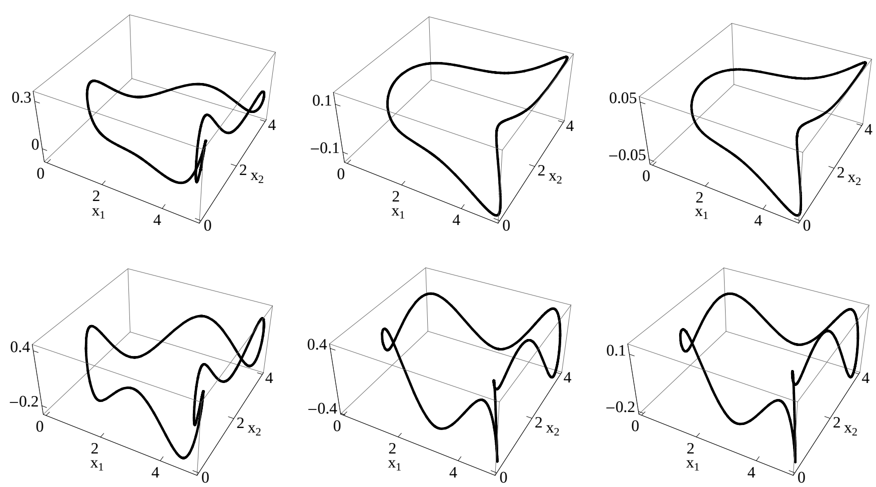

Fig. 1.2 Graphs of the $\Phi_{\alpha i}$.

As a second example, consider the 'expanding' ellipse ∂S of parametric equations

$$x_1(t) = 2k\cos(\pi t),$$
$$x_2(t) = k\sin(\pi t), \quad 0 \le t \le 2.$$

The graph of $\det \mathscr{C}$ as a function of k is shown in Fig. 1.3.

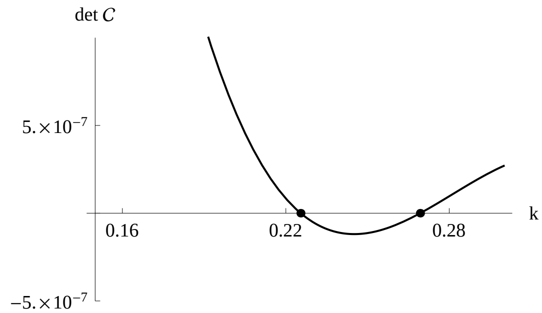

Fig. 1.3 Graph of $\det\mathscr{C}$.

Here, $\det\mathscr{C} = 0$ for $k = 0.22546$ and $k = 0.26934$.

References

[Co94] Constanda, C.: On integral solutions of the equations of thin plates. *Proc. Roy. Soc. London Ser. A*, **444**, 261–268 (1994).

[Co00] Constanda, C.: *Direct and Indirect Boundary Integral Equation Methods*, Chapman & Hall/CRC, Boca Raton, FL (2000).

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